Chapter 9

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Economic Core, Fair Allocations, and Social Choice Theory

9.1 Introduction

In this chapter, we briefly discuss some topics in the framework of general equilibrium theory, namely economic core, fair allocations, and social choice theory. The theory of core is important because it gives an insight into how a competitive equilibrium is achieved as a result of individual strategic behavior instead of results of an auctioneer and the Walrasian tatonnement mechanism. It shows the necessity of adopting a market institution as long as individuals behave self-interestedly.

We have also seen that Pareto optimality may be too weak a criterion to be meaningful. It does not address any question about income distribution and equity of allocations. Fairness is a notion to overcome this difficulty. This is one way to restrict a set of Pareto optimum.

In a slightly different framework, suppose that a society is deciding the social priority among finite alternatives. Alternatives may be different from Pareto optimal allocations. Let us think of a social “rule” to construct the social ordering (social welfare function) from many individual orderings of different alternatives. The question is: is it possible to construct a rule satisfying several desirable properties? Both “fairness” and “social welfare function” address a question of social justice.

9.2 The Core of Exchange Economies

The use of a competitive (market) system is just one way to allocate resources. What if we use some other social institution? Would we still end up with an allocation that was “close” to a competitive equilibrium allocation? The answer will be that, if we allow agents to form coalitions, the resulting allocation can only be a competitive equilibrium allocation when the economy becomes large. Such an allocation is called a core allocation and was originally considered by Edgeworth.

The core is a concept in which every individual and every group agree to accept an allocation instead of moving away from the social coalition. There is some reason to think that the core is a
meaningful political concept. If a group of people find themselves able, using their own resources to achieve a better life, it is not unreasonable to suppose that they will try to enforce this threat against the rest of the community. They may find themselves frustrated if the rest of the community resorts to violence or force to prevent them from withdrawing.

The theory of the core is distinguished by its parsimony. Its conceptual apparatus does not appeal to any specific trading mechanism, nor does it assume any particular institutional setup. Informally, the notion of competition that the theory explores is one in which traders are well informed of the economic characteristics of other traders, and in which the members of any group of traders can bind themselves to any mutually advantageous agreement.

For simplicity, we consider exchange economies. We say two agents are of the same type if they have the same preferences and endowments. The $r$-replication of the original economy: there are $r$ times as many agents of each type in the original economy.

**Definition. 9.2.1 (Blocking Coalition).** A group of agents $S$ (a coalition) is said to block (improve upon) a given allocation $x$ if there is a Pareto improvement with their own resources, that is, if there is some allocation $x'$ such that

a. it is feasible for $S$, i.e., $\sum_{i \in S} x'_i \leq \sum_{i \in S} w_i$

b. $x'_i \succeq_i x_i$ for all $i \in S$ and $x'_k \succ_k x_k$ for some $k \in S$.

A coalition is a group of agents, and thus it is a subset of $n$ agents. If the original economy has $N$ agents, then $2^N - 1$ possible coalitions can be formed.

**Definition. 9.2.3 (Core).** A feasible allocation $x$ is said to have the core property if it cannot be improved upon for any coalition. The core of an economy is the set of all feasible allocations that have the core property, i.e., it is the set of allocations that have no blocking coalitions.

**Remark 9.2.1.** Every allocation in the core is Pareto optimal (coalition by whole people), but not every Pareto optimal allocation is in the core. So, the set of core allocations is a subset of the set of Pareto efficient allocations.

**Definition. 9.2.3 (Individual Rationality).** An allocation $x$ is individually rational if $x_i \succeq_i w_i$ for all $i = 1, \ldots, n$.

The individual rationality condition is also called the participation condition which means that a person will not participate in the economic activity if he is worse off than at the initial endowment.

**Remark 9.2.2.** Every allocation in the core must be individually rational.

**Remark 9.2.3.** When $n = 2$ and preference relations are weakly monotonic, an allocation is in the core if and only if it is Pareto optimal and individually rational.

**Remark 9.2.4.** Even though a Pareto optimal allocation is independent of individual endowments, an allocation in the core depends on individual endowments.

What is the relationship between core allocations and competitive equilibrium allocations?
Theorem. 9.2.1. Under local non-satiation, if \((x, p)\) is a competitive equilibrium, then \(x\) has the core property.

Proof. Let \(\succeq_j\) satisfy local non-satiation. Assume \((x, p)\) is a competitive equilibrium. Suppose for contradiction that \(x\) does not have the core property. Then, there exists a group of agents \(S\) and an allocation \(x'\) such that \(\sum_{i \in S} x'_i \leq \sum_{i \in S} w_i\) and \(x'_i \succeq_j x_i\) for all \(i \in S\) and \(x'_k \succ_k x_k\) for some \(k \in S\). By local non-satiation, \(\forall i \in S: p \cdot x'_i \geq p \cdot x_i\). By utility maximization, \(\exists k \in S: p \cdot x'_k > p \cdot x_k\). Taking summation, we have
\[
\sum_{i \in S} p \cdot x'_i > \sum_{i \in S} p \cdot x_i = \sum_{i \in S} p \cdot w_i
\]
However, this contradicts the fact that \(x'\) is feasible for \(S\). Therefore, the competitive equilibrium must be an allocation in the core.

We refer to the allocations in which consumers of the same type get the same consumption bundles as equal-treatment allocations. It can be shown that any allocation in the core must be an equal-treatment allocation.

Proposition. 9.2.1 (Equal Treatment in the Core). Suppose agents’ preferences are strictly convex. If \(x\) is an allocation in the \(r\)-core of a given economy, then any two agents of the same type must receive the same bundle.

Proof. Assume preferences are strictly convex. Let \(x\) be an allocation in the core and index the \(2r\) agents using subscripts \(A_1, \ldots, A_r\) and \(B_1, \ldots, B_r\). Suppose for contradiction that all agents of the same type do not get the same allocation. Then, there will be at least one person of each type who does strictly worse than everyone else. Denote the allocations of these two individuals as \(x_{uA}\) and \(x_{uB}\). If there are ties, select any of the tied agents.

Let \(\bar{x}_A = \frac{1}{r} \sum_{i=1}^r x_{A_i}\) and \(\bar{x}_B = \frac{1}{r} \sum_{i=1}^r x_{B_i}\) be the average bundle of the type-\(A\) and type-\(B\) agents. Since the allocation \(x\) is feasible and every agent of the same type gets the same endowment, we have
\[
\bar{x}_A + \bar{x}_B = \frac{1}{r} \sum_{i=1}^r x_{A_i} + \frac{1}{r} \sum_{i=1}^r x_{B_i} = \frac{1}{r} \sum_{i=1}^r w_{A_i} + \frac{1}{r} \sum_{i=1}^r w_{B_i} = \frac{1}{r} rw_{A} + \frac{1}{r} rw_{B} = w_A + w_B
\]
So, \((\bar{x}_A, \bar{x}_B)\) is feasible for the coalition consisting of the two underdogs. Remember, we are assuming that at least one of the underdogs receives a different bundle than the rest of the agents of the same type. Without loss of generality, let the \(A\) underdog receive a different bundle than the rest of the \(A\) agents. Since \(\bar{x}_A\) is the weighted average of bundles that are at least as good as \(x_{uA}\), then underdog \(A\) strictly prefers \(\bar{x}_A\) to \(x_{uA}\) by strict convexity of preferences. Moreover, the \(B\) underdog finds \(\bar{x}_B\) at least as good as \(x_{uB}\). Thus, underdog \(A\) and underdog \(B\) can form a coalition that can impose upon the allocation \(x\). However, this contradicts the fact that \(x\) has the core property.

Since any allocation in the core must award agents of the same type with the same bundle, we can examine the cores of replicated two-agent economies by use of the Edgeworth box diagram. Instead of a point \(x\) in the core representing how much \(A\) gets and how much \(B\) gets, we can think
of $x$ as telling us how much each agent of type $A$ gets and how much each agent of type $B$ gets. The above proposition tells us that all points in the $r$-core can be represented in this manner.

The following theorem is a converse of Theorem 9.2.1 and shows that any allocation that is not a market equilibrium allocation must eventually not be in the $r$-core of the economy. This means that core allocations in large economies look just like Walrasian equilibria.

**Theorem. 9.2.2 (Shrinking Core Theorem).** Suppose $\succeq_i$ are strictly convex and continuous. Suppose $x^*$ is a unique competitive equilibrium allocation. If $y$ is not a competitive equilibrium, then there is some replication $V$ such that $y$ is not in the $V$-core.

**Proof.** Let $\succeq_i$ be strictly convex and continuous. Suppose $x^*$ is a unique competitive equilibrium. Assume $y$ is not a competitive equilibrium allocation. We want to show that there is a coalition such that the point $y$ can be improved upon for $V$-replication.

Since $y$ is not a competitive equilibrium, the line segment through $y$ and $w$ must cut at least one agent’s indifference curve through $y$. If the line segment was tangent to both indifference curves at $y$, then $y$ would be a competitive equilibrium. Without loss of generality, let the line segment cut through agent $A$’s indifference curve through $y$. So, by strict convexity and continuity of $\succeq_i$, there are integers $V$ and $T$ with $0 < T < V$ such that

$$g_A \equiv \frac{T}{V}w_A + (1 - \frac{T}{V})y_A \succ_A y_A$$

Now, form a coalition consisting of $V$ consumers of type $A$ and $V - T$ consumers of type $B$ and consider the allocation $x = (g_A, \ldots, g_A, y_B, \ldots, y_B)$. We want to show that $x$ is feasible for this coalition.

$$Vg_A + (V - T)y_B = V \left[ \frac{T}{V}w_A + (1 - \frac{T}{V})y_A \right] + (V - T)y_B$$

$$= Tw_A + Vy_A - Ty_A + Vy_B - Ty_B$$

$$= Tw_A + (V - T)y_A + (V - T)y_B$$

$$= Tw_A + (V - T)(y_A + y_B)$$

$$= Tw_A + (V - T)(w_A + w_B)$$

$$= Vw_A + (V - T)w_B$$

Important to note, since $y$ is in the core before replication, then it is Pareto optimal, and hence, feasible. So, $y_A + y_B = w_A + w_B$. Thus, $x$ is feasible in the coalition and $g_A \succ_A y_A$ for all agents in type $A$ and $y_B \sim_B y_B$ for all agents in type $B$. Therefore, $y$ is not in the $V$-core for the $V$-replication of the economy. $\Box$

**Remark 9.2.5.** Thus, as we increase the $V$ replication of the economy, any point originally in the core that’s not a competitive equilibrium will, eventually, no longer be in the core as the economy grows. The shrinking core theorem then shows that the only allocations that are in the core of a large economy are market equilibrium allocations, and thus Walrasian equilibria are robust: even very weak equilibrium concepts, like that of the core, tend to yield allocations that are close to
Walrasian equilibria for larger economies. Thus, this theorem shows the essential importance of competition and full economic freedom. 

Remark 9.2.6. Many of the restrictive assumptions in this proposition can be relaxed such as strict monotonicity, convexity, uniqueness of competitive equilibrium, and two types of agents.

Theorem. 9.2.3 (Limit Theorem on the Core). Under strict convexity and continuity, the core of a replicated two person economy shrinks when the number of agents for each type increases, and the core coincides with the competitive equilibrium allocation if the number of agents goes to infinity.

This result means that any allocation which is not a competitive equilibrium allocation is not in the core for some \( r \)-replication.

9.3 Fairness of Allocations

Definition. 9.3.1 (Envy). An agent \( i \) is said to envy agent \( k \) if agent \( i \) prefers agent \( k \)'s consumption, i.e., \( x_k \succ_i x_i \).

Pareto efficiency gives a criterion of how the goods are allocated efficiently, but it may be too weak a criterion to be meaningful. It does not address any questions about income distribution, and does not give any “equity” implication.

Definition. 9.3.1. (Envy). An agent \( i \) is said to envy agent \( k \) if agent \( i \) prefers agent \( k \)'s consumption, i.e., \( x_k \succ_i x_i \).

What is the equitable allocation? How can we define the notion of an equitable allocation?

Definition. 9.3.2 (Envy-free, also known as Equitable). An allocation \( x \) is envy-free, or equitable, if no one envies anyone else, i.e., \( \forall i \in N, \forall k \in N : x_i \succeq_i x_k \).

Fairness is a notion that may overcome the difficulty just described. This is one way to restrict the whole set of Pareto efficient outcomes to a small set of Pareto efficient outcomes that satisfy the other properties.

Definition. 9.3.3 (Fairness). An allocation \( x \) is said to be fair if it is both Pareto optimal and equitable.

Remark 9.3.1. By definition, the set of fair allocations is a subset of the set of Pareto efficient allocations. Therefore, fairness restricts the size of Pareto optimal allocations.

Definition. 9.3.4 (Strictly Envy-Free or Strictly Equitably). An allocation \( x \) is strictly equitable, or strictly envy-free, if no one envies any other coalition.

An agent \( i \) envies a coalition \( S (i \notin S) \) at an allocation \( x \) if \( x_S \succ_i x_i \), where \( x_S = \frac{1}{|S|} \sum_{j \in S} x_j \).
Definition. 9.3.4 (Strict Fairness). An allocation $x$ is said to be strictly fair if it is both Pareto optimal and strictly equitable.

Remark 9.3.2. If an allocation is strictly envy-free, then it is envy-free. The set of strictly fair allocations are a subset of the set of Pareto optimal allocations.

Remark 9.3.3. For a two person exchange economy, if $x$ is Pareto optimal, then it is impossible for both persons to envy each other.

Remark 9.3.4. It is clear that every strictly fair allocation is a fair allocation, but the converse may not be true. However, when $n = 2$, a fair allocation is a strictly fair allocation.

How to test a fair allocation? Let us restrict an economy to a two-person economy. An easy way for agent $A$ to compare his own allocation $x_A$ with agent $B$’s allocation $x_B$ in the Edgeworth Box is to find a point symmetric of $x_A$ against the center of the box. That is, draw a line from $x_A$ to the center of the box and extrapolate it to the other side by the same length to find $x'_A$, and then make the comparison. If the indifference curve through $x_A$ cuts “below” $x'_A$ then $A$ envies $B$. So, we have the following way to test whether an allocation is a fair allocation:

a. Is it Pareto optimal? If the answer is “yes,” then go to step 2. If not, then stop.

b. Construct a reflection point $(x_B, x_A)$. Not that $\frac{x_A + x_B}{2}$ is the center of the Edgeworth Box.

c. Compare $x_B$ with $x_A$ for person $A$ to see if $x_B \succ_A x_A$ and compare $x_A$ with $x_B$ for person $B$ to see if $x_A \succ_B x_B$. If the answer is “no” for both persons, then it is a fair allocation.

We have given some desirable properties of “fair” allocations. A question is whether it exists at all. The following theorem provides one sufficient condition to guarantee the existence of fairness.

Theorem. 9.3.1. Let $(x^*, p^*)$ be a competitive equilibrium. Under local non-satiation, if all individuals’ income is the same, i.e., $p^* \cdot w_1 = \cdots = p^* \cdot w_n$, then $x^*$ is a strictly fair allocation.

Proof. By local non-satiation, $x^*$ is Pareto efficient. Now, we want to show $x^*$ is strictly equitable. Suppose for contradiction that it is not. Then, there exists an agent $i$ and a coalition $S$ with $i \notin S$ such that

$$\frac{1}{|S|} \sum_{j \in S} x^*_j \succ_i x^*_i$$

By utility maximizaiton and local non-satiation, we have that

$$\frac{1}{|S|} \sum_{j \in S} p \cdot x^*_j > p^* \cdot x^*_i = p^* \cdot w_i$$

However, this contradicts the fact that

$$\frac{1}{|S|} \sum_{j \in S} p^* \cdot x^*_j = \frac{1}{|S|} \sum_{j \in S} p^* \cdot w_j = p^* \cdot w_i$$

since $p^* \cdot w_1 = \cdots = p^* \cdot w_n$. Therefore, $x^*$ must be a strictly fair allocation. \qed
Agents have an initial endowment \( w_i \) and then endowments are redistributed using transfer payments in such a way that income is the same for all individuals. After the redistribution, trading begins and we obtain a competitive equilibrium allocation, which will be strictly fair.

**Definition. 9.3.6.** An allocation \( x \in \mathbb{R}^{nL}_+ \) is an equal income Walrasian allocation if there exists a price vector \( p \) such that

a. \( p \cdot x_i \leq p \cdot \bar{w} \), where \( \bar{w} = \frac{1}{n} \sum_{i=1}^{n} w_i \) (average endowment)

b. \( x'_i \succ_i x_i \) implies \( p \cdot x'_i > p \cdot \bar{w} \)

c. \( \sum x_i \leq \sum w_i \)

Notice that every equal income Walrasian allocation \( x \) is a competitive equilibrium allocation with \( w_i = \bar{w} \) for all \( i \).

**Corollary. 9.3.1.** Under local non-satiation, every equal income Walrasian allocation is a strictly fair allocation.

**Remark 9.3.5.** An allocation that is an “equal” division of resources does not give “fairness,” but trading from “equal” position will result in a “fair” allocation. This “divide-and-choose” recommendation implies that if the center of the box is chosen as the initial endowment point, the competitive equilibrium allocation is fair. A political implication of this remark is straightforward. Consumption of equal bundles is not Pareto optimal if preferences are different. However, an equal division of endowment plus competitive markets result in fair allocations.

**Remark 9.3.6.** A competitive equilibrium from an equitable (but not “equal” division) endowment is not necessarily fair.

**Remark 9.3.7.** Fair allocations are defined without reference to initial endowments. Since we are dealing with optimality concepts, initial endowments can be redistributed among agents in a society. Agents have an initial endowment \( w_i \), and then it is redistributed using transfer payments such that \( \forall \ i \in N : w_i = \bar{w} \). After the redistribution, trading begins and we obtain a competitive equilibrium allocation, which will be strictly fair.

In general, there is no relationship between an allocation in the core and a fair allocation. However, when the social endowments are divided equally among two persons, we have the following theorem.

**Theorem. 9.3.2.** In a two-person exchange economy, if \( \succsim_i \) are convex, and if the total endowments are equally divided among individuals, then the set of allocations in the core is a subset of (strictly) fair allocations.

**Proof.** We know that an allocation \( x \) in the core is Pareto optimal. We only need to show that \( x \) is equitable. Since \( x \) is a core allocation, then it is individually rational. Suppose for contradiction that \( x \) is not equitable. Then, there exists some agent \( i \), say agent \( A \), such that

\[
x_B \succ_A x_A \succsim_A w_A = \frac{1}{2} (w_A + w_B) = \frac{1}{2} (x_A + x_B)
\]
by noting that \( w_A = w_B \) and \( x \) is feasible. Thus \( x_A \succeq_A \frac{1}{2}(x_A + x_B) \). But, on the other hand, since \( \succeq_A \) is convex, then \( x_B \succ_A x_A \) implies that \( \frac{1}{2}(x_A + x_B) \succ_A x_A \), which is a contradiction.

### 9.4 Social Choice Theory

#### 9.4.1 Introduction

In this section, we present a very brief summary and introduction of social choice theory. We analyze the extent to which individual preferences can be aggregated into social preferences, or more directly into social decisions, in a “satisfactory” manner (in a manner compatible with the fulfilment of a variety of desirable conditions).

As was shown in the discussion of “fairness,” it is difficult to come up with a criterion (or a constitution) that determines that society’s choice. Social choice theory aims at constructing such a rule, which could be allied with not only Pareto efficient allocations, but also any alternative that a society faces. We will give some fundamental results of social choice theory: Arrow’s Impossibility Theorem, which states there does not exists any non-dictatorial social welfare function satisfying a number of “reasonable” assumptions.

#### 9.4.2 Basic Settings

To simplify notation, \( aP_i b \) means that \( i \) strictly prefers \( a \) to \( b \). We assume that the individual preferences are all strict, in other words, that there is never a situation of indifference. For all \( i, a \) and \( b \), one gets preference \( aP_i b \) or \( bP_i a \). This hypothesis is not very restrictive if \( X \) is finite.

\[ N = \{1, \ldots, n\} \] the set of individuals
\[ X = \{x_1, \ldots, x_m\}, \ (m \geq 3) \] the set of alternatives (outcomes)
\[ P_i = \succ_i \] the strict preference orderings of agent \( i \)
\[ \mathcal{P}_i \] the class of allowed individual orderings; \( P_i \in \mathcal{P}_i \)
\[ P = (P_1, \ldots, P_n) \] a preference ordering profile
\[ \mathcal{P} \] the set of all profiles of individual orderings; \( P \in \mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_n \)
\[ S(X) \] the class of allowed social orderings.

Arrow’s social welfare function:
\[ F : \mathcal{P} \rightarrow S(X) \]
which is a mapping from individual ordering profiles to social orderings

Gibbard-Satterthwaite’s social choice function (SCF) is a mapping from individual preference orderings to the alternatives
\[ f : \mathcal{P} \rightarrow X \]

Note that even though individuals’ preference orderings are transitive, a social preference ordering may not be transitive. To see this, consider the following example.
Example. 9.4.1. (The Condorcet Paradox). Suppose a social choice is determined by the majority voting rule. Does this determine a well-defined social welfare function? The answer is in general no by the well-known Condorcet paradox. Consider a society with three agents and three alternatives: \( x, y, z \). Suppose each person’s preferences is given by:

\[
\begin{align*}
    x &\succ_1 y \succ_1 z \quad \text{(by person 1)} \\
    y &\succ_2 z \succ_2 x \quad \text{(by person 2)} \\
    z &\succ_3 x \succ_3 y \quad \text{(by person 3)}
\end{align*}
\]

By the majority rule,

For \( x \) and \( y \), \( xFy \) (by social preference)

For \( y \) and \( z \), \( yFz \) (by social preference)

For \( x \) and \( z \), \( zFx \) (by social preference)

Then pairwise majority voting tells us that \( x \) must be socially preferred to \( y \), \( y \) must be socially preferred to \( z \), and \( z \) must be socially preferred to \( x \). This cyclic pattern means that social preference is not transitive.

The number of preference profiles can increase very rapidly with increase in the number of alternatives.

Example. 9.4.2. \( X = \{x, y, z\}, n = 3 \).

\[
\begin{align*}
    x &\succ y \succ z \\
    x &\succ z \succ y \\
    y &\succ x \succ z \\
    y &\succ z \succ x \\
    z &\succ x \succ y \\
    z &\succ y \succ x
\end{align*}
\]

Thus, there are six possible individual orderings, i.e., \( |P_i| = 6 \), and therefore there are \( |P_1| \times |P_2| \times |P_3| = 216 \) possible combinations of 3-individual preference orderings on three alternatives. The social welfare function is a mapping from each of these 216 entries (cases) to one particular social ordering (among six possible social orderings of three alternatives). The social choice function is a mapping from each of these 216 cases to one particular choice (among three alternatives). A question we will investigate is what kinds of desirable conditions should be imposed on these social welfare or choice functions.

You may think of a hypothetical case of sending a letting to your congressional representative that lists your preference orderings, \( P_i \), on announced national projections (alternatives), such as reducing deficits, expanding the medical program, reducing the social security program, increasing the national defense budget, etc. Congress convenes with a huge stake of letters \( P \) and tries to come up with national priorities \( F(P) \). You want Congress to make a certain rule (i.e., the Constitution) to form national priorities out of individual preference orderings. This is a question addressed in social choice theory.
9.4.3 Arrow’s Impossibility Theorem

Unrestricted Domain (UD):

Definition. 9.4.1. A class of allowed individual orderings \( \mathcal{P} \) consists of all possible orderings defined on \( X \).

Pareto Principle (P):

Definition. 9.4.2. If for \( x, y \in X \), \( xP_i y \) for all \( i \in N \), then \( xF(P)y \) (social preferences).

Independence of Irrelevant Alternatives (IIA):

Definition. 9.4.3. For any two alternatives \( x, y \in X \) and any two preference profiles \( P, P' \in \mathcal{P} \), \( xF(p)y \) and \( \{i : xP_i y\} = \{i : xP'_i y\} \) for all \( i \) implies that \( xF(P')y \). That is, the social preference between any two alternatives depends only on the profile of individual preferences over the same alternatives.

Remark 9.4.1. IIA means that the ranking between \( x \) and \( y \) for any agent is equivalent in terms of \( P \) and \( P' \) implies that the social ranking between \( x \) and \( y \) by \( F(P) \) and \( F(P') \) is the same. In other words, if two different preference profiles are the same on \( x \) and \( y \), then the social order must be the same on \( x \) and \( y \).

Remark 9.4.2. By IIA, any change in preference ordering other than the ordering of \( x \) and \( y \) should not affect social ordering between \( x \) and \( y \).

Definition. 9.4.4 (Dictator). There is some agent \( i \in N \) such that \( F(P) = P_i \) for all \( P \in \mathcal{P}^n \), and agent \( i \) is called the dictator.

Theorem. 9.4.1 (Arrow’s Impossibility Theorem). Any social welfare function that satisfies \( m \geq 3 \), UD, P, and IIA conditions is dictatorial.

The impact of Arrow’s Impossibility Theorem has been quite substantial. Obviously, the result of the theorem is a disappointment. The most pessimistic reaction to it is to conclude that there is just no acceptable way to aggregate individual preferences, and hence, no theoretical basis for treating welfare issues. A more moderate reaction, however, is to examine each of the assumptions of the theorem to see which might be given up. Conditions imposed on social welfare functions may be too restrictive. Indeed, when some conditions are relaxed, then the results could be positive, e.g., UD is usually relaxed.

9.4.4 Some Positive Result: Restricted Domain

When some of the assumptions imposed in Arrow’s Impossibility Theorem are removed, the result may be positive. For instance, if alternatives have certain characteristics which could be placed in a spectrum, preferences may show some patterns and may not exhaust all possible orderings on \( X \), thus violated UD. In the following setup, we consider the case of a restricted domain. A famous example is a class of “single-peaked” preferences. We show that under the assumption of single-peaked preferences, non-dictatorial aggregation is possible.
Definition. 9.4.5. A binary relation \( \geq \) on \( X \) is a linear order if \( \geq \) is reflexive \((x \geq x)\), transitive \((x \geq y \geq z \implies x \geq z)\), and total (for distinct \( x, y \in X : x \geq y \lor y \geq x \), but not both).

Example. \( X = \mathbb{R} \) and \( x \geq y \).

Definition. 9.4.6. \( \succsim_i \) is said to be single-peaked with respect to the linear order \( \geq \) on \( X \), if there is an alternative \( x \in X \) such that \( \succsim_i \) is increasing with respect to \( \geq \) on the lower contour set \( L(x) = \{y \in X : y \leq x\} \) and decreasing with respect to \( \geq \) on the upper contour set \( U(x) = \{y \in X : y \geq x\} \). That is

\[
\text{a. } x \geq z > y \implies z \succsim_i y \\
\text{b. } y > z \geq x \implies z \succsim_i y
\]

In words, there is an alternative that represents a peak of satisfaction. Moreover, satisfaction increases as we approach this peak so that, in particular, there cannot be any other peak of satisfaction.

Given a profile of preference \((\succsim_1, \ldots, \succsim_n)\), let \( x_i \) be the maximal alternative for \( \succsim_i \) (we will say that \( x_i \) is “individual \( i \)’s peak”).

Definition. 9.4.7. Agent \( h \in N \) is a median agent for the profile \((\succsim_1, \ldots, \succsim_n)\) if \( \#\{i \in N : x_i \geq x_h\} \geq \frac{n}{2} \) and \( \#\{i \in N : x_h \geq x_i\} \geq \frac{n}{2} \).

Proposition. 9.4.1. Suppose that \( \geq \) is a linear order on \( X \) and \( \succsim_i \) is single-peaked. Let \( h \in N \) be a median agent, then the majority rule \( \tilde{F}(\succsim) \) is aggregatable:

\[
\forall y \in X : x_h \tilde{F}(\succsim) y
\]

Proof. Take any \( y \in X \) and suppose that \( x_h > y \) (the argument is symmetric for \( y > x_h \)). We need to show that

\[
\#\{i \in N : x_h \succsim_i y\} \geq \#\{i \in N : y \succsim_i x_h\}
\]

Consider the set of agents \( S \subset N \) that have peaks larger or equal to \( x_h \), i.e., \( S = \#\{i \in N : x_i \geq x_h\} \). Then, \( \forall i \in S : x_i \geq x_h > y \). Hence, by single-peakness of \( \succsim_i \) with respect to \( \geq \), \( \forall i \in S : x_h \succsim_i y \). So, \( \#S \leq \#\{i \in N : x_h \succsim_i y\} \). On the other hand, because \( h \) is the median agent, we have that \( \#S \geq \frac{n}{2} \) and so

\[
\#\{i \in N : y \succsim_i x_h\} \leq \#(N \setminus S) \leq \frac{n}{2} \leq \#S \leq \#\{i \in N : x_h \succsim_i y\}
\]

So, the peak \( x_h \) of the median agent is socially optimal, i.e., it cannot be defeated by any other alternative, by majority voting. Any alternative having this property is called a Condorcet winner. Therefore, a Condorcet winner exists whenever the preferences of all agents are single-peaked with respect to the same linear order.